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FINITE EMBEDDING THEOREMS FOR PARTIAL PAIRWISE BALANCED DESIGNS

William B. POUCHER

Dept. of Mathematics, David Lipscomb College, Nashville, Tennessee 37203, U.S.A.

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The pair (P, p) is a (partial) (n, b)-PBD if (P, p) is a (partial) pairwise balanced design with the property that $|P| = n$ and each block in p has exactly b elements. The following theorems are proved.

Theorem. *If (P, p) is an (n, b)-PBD and $n > b \geq 4$, then (P, p) has an isomorphic disjoint mate.* (Theorem 2.3)

Theorem. *Suppose k and b are positive integers and $b \geq 5$. There is a constant $C(k, b)$ such that if (P, p) is an (n, b)-PBD and $n > C(k, b)$, then there exist k mutually disjoint isomorphic mates of (P, p).* (Theorem 2.2)

Theorem. *Suppose k and b are positive integers, $k \geq 2$ and $b \geq 5$. If $(P, p_1), (P, p_2), \ldots, (P, p_k)$ is a collection of partial $(|P|, b)$-PBD's, there exist k (n, b)-PBD's $(X, x_1), (X, x_2), \ldots, (X, x_k)$ such that (P, p_i) is embedded in (X, x_i) and for $i \neq j$, $p_i \cap p_j = x_i \cap x_j$. Additionally the existence of certain collections valuable in embedding is explored.* (Theorem 4.10)

1. Introduction

A *pairwise balanced design* (PBD) is a pair (P, p) where P is a finite set and p is a collection of subsets of P called *blocks* such that

(i) each block in p contains at least two points, and

(ii) every two element subset of P is contained in exactly one block of p.

If (P, p) satisfies condition (i) and the weaker condition:

(ii)* every two element subset of P is contained in at most one block of p, then (P, p) is called a *partial* PBD. The number $|P|$ is called the order of the (partial) PBD (P, p). The (partial) PBD (P, p) is called a (partial) (n, b)-PBD if and only if P has order n and each block in p contains exactly b elements. Two (n, b)-PBD's, (P, p_1) and (P, p_2) are said to be disjoint provided that p_1 and p_2 have no blocks in common. The (partial) PBD (P, p) is said to be *embedded* in the (partial) PBD (Q, t) provided that $P \subseteq Q$ and $p \subseteq t$. It is well known that in order that there exist an (n, b)-PBD, that $n - 1 \equiv 0 \bmod (b - 1)$ and $n(n-1) \equiv 0 \bmod (b(b-1))$ [1]. In fact Wilson in [2] proved that if $b \geq 2$, then there exists a constant $c(b)$ such that if

$n > c(b)$ and $n-1 \equiv 0 \pmod{b-1}$ and $n(n-1) \equiv 0 \pmod{b(b-1)}$, then there exists an (n, b)-PBD. A *Steiner triple system* is, of course, an $(n, 3)$-PBD. In [3], Treash showed that a finite partial Steiner triple system can be embedded in a Steiner triple system. In [4], Lindner proved that any pair of finite disjoint partial Steiner triple systems can be embedded in a pair of finite disjoint Steiner triple systems. Recently, Lindner and Rosa in [5] proved the best possible result along these lines: that is if $(P, p_1), (P, p_2), \ldots, (P, p_k)$ is any collection of finite partial Steiner triple systems then there exists finite Steiner triple systems $(X, x_1), (X, x_2), \ldots, (X, x_k)$ such that (P, p_i) is embedded in (X, x_i) and $x_i \cap x_j = p_i \cap p_j$ for all $i \neq j$. The purpose of this paper is to generalize this result to (n, b)-PBD's for any $b \geq 5$. Additionally, it is shown that every (n, b)-PBD has an isomorphic disjoint mate for all $b \geq 4$ and $n > b$.

2. Isomorphic disjoint (n, b)-PBD's

The following lemma and theorem prove the existence of k disjoint isomorphic mates for each (n, b)-PBD where $b \geq 5$ and n is larger than some constant determined by k and b.

Lemma 2.1. *Suppose k and b are positive integers, $b \geq 5$. There exists an integer $C(k, b)$ such that if $n > C(k, b)$, then $k(M(n, b)) < n!$ where $M(n, b) = (n(n-1)/(b(b-1)))^2\,(n-b)!\,b!$.*

Proof. Take $C(k, b) = (kb!/(b(b-1))) + 4$.

Theorem 2.2. *Suppose k and b are positive integers, $b \geq 5$. If (P, p) is an (n, b)-PBD and $n > C(k, b)$, then there exist $k+1$ permutations on P, $\alpha_0 = i, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k$ such that $(P, p\alpha_0 = p), (P, p\alpha_1), \ldots, (P, p\alpha_k)$ are $k+1$ mutually disjoint (n, b)-PBD's where $p\alpha_i$ is the collection of blocks obtained from p by applying the permutation α_i to the elements of P.*

Note that α_i is an isomorphism from (P, p) to $(P, p\alpha_i)$. Furthermore, there are $n!$ permutations on P and that

$$(n(n-1)/(b(b-1)))^2(n-b)!\,b! = M(n, b)$$

is a poor but usable upperbound on the number of permutations that map some block of p onto a block of p' where (P, p) and (P, p') are (n, b)-PBD's. This follows from the fact that each ordered pair (x, y) in $p \times p'$ can be associated with the $b!(n-b)!$ permutations of P that map x to y. Since $|p \times p'| = (n(n-1)/(b(b-1)))^2$ there are at most $(n(n-1)/(b(b-1)))^2(n-b)!\,b! = M(n, b)$ permutations of the desired type.

Proof. The proof follows by induction on k. Suppose (P, p) is an (n, b)-PBD and $n > C(k, b)$. Thus $M(n, b) < n!$. Let α_0 be the identity map on P. There are exactly $M(n, b)$ permutations on P that map a block of p onto a block of p. Since $k \geq 1$, $M(n, b) < n!$ and there is a permutation α_1 on P such that (P, p) and $(P, p\alpha_1)$ are disjoint. Since there are exactly $M(n, b)$ permutations on P that map a block of p onto a block of $p\alpha_1$, there are at most $2M(n, b)$ permutations on P that map a block of p onto a block of $p \cup p\alpha_1$. If $k \geq 2$, then $2M(n, b) < n!$ so there is a permutation α_2 on P that does not map a block of p onto a block of $p \cup p\alpha_1$. Now (P, p), $(P, p\alpha_1)$ and $(P, p\alpha_2)$ are mutually disjoint. Successive α_j can be chosen in a like manner based on the selection of $\alpha_0, \alpha_1, \ldots, \alpha_{j-1}$ until $j(M(n, b)) < n!$ is no longer a true statement. Thus the necessary $\alpha_0 = i, \alpha_1, \alpha_2, \ldots, \alpha_k$ can be found.

Theorem 2.3. *If (P, p) is an (n, b)-PBD and $n > b \geq 4$, then (P, p) has an isomorphic disjoint mate.*

Proof. Suppose $n > b \geq 4$. Now

$$n!/((n-b)!\,b!) = (n/b)((n-1)/(b-1))((n-2)/(b-2))\cdots((n-(b-1))/1).$$

For $0 \leq i \leq b-1$, $(n-i)/(b-i) > 1$ so

$$(n/b)((n-1)/(b-1))((n-2)/(b-2))((n-3)/(b-3)) \leq n!/((n-b)!\,b!).$$

Since $n/b < (n-2)/(b-2)$ and $(n-1)/(b-1) < (n-3)/(b-3)$,

$$(n/b)((n-1)/(b-1))(n/b)((n-1)/(b-1)) < n!/((n-b)!b!).$$

So $(n(n-1)/(b(b-1)))^2(n-b)!\,b! < n!$ and $M(n, b) < n!$ Thus by an argument for $k = 1$ in Theorem 2.2 we get the necessary permutation α_1 such that (P, p) and $(P, p\alpha_1)$ are mutually disjoint where (P, p) is an (n, b)-PBD.

3. Separating sets

We begin this section with some definitions.

Definition 3.1. Column (i, j) of an orthogonal array A is the unique column whose cell on row 1 is occupied by i and whose cell on row 2 is occupied by j. (See [1] for the definition of orthogonal array.)

Definition 3.2. The $b \times n^2$ orthogonal arrays A and B are said to be *disjoint on column* (i, j) provided that for some integer $m > 2$, the cells in row m of column (i, j) of A and B are occupied by different symbols.

Definition 3.3. The $b \times n^2$ orthogonal arrays A and B are said to be disjoint provided that they are disjoint on every column.

Definition 3.4. L is a separating set of dimension $b \times n^2$ and strength k if and only if $L = \{L(i,j) \mid 1 \leq i \leq 2, 1 \leq j \leq k\}$ is a collection of $2k$ orthogonal arrays each of dimension $b \times n^2$ and having the following properties:

(i) For every i, $1 \leq i \leq k$, every cell of column $(1,1)$ of $L(1,i)$ is occupied by 1.

(ii) For every $i, j \in \{1,2,\ldots,k\}$, $i \neq j$, $L(1,i)$ and $L(1,j)$ are disjoint on each column except column $(1,1)$.

(iii) For every $i, j \in \{1,2,\ldots,k\}$, $i \neq j$, and $m = 1$ or 2, $L(2,i)$ and $L(m,j)$ are disjoint.

Theorem 3.5. *If there exists a $b \times n^2$ orthogonal array where $b \geq 5$, then there exists a separating set of dimension $b \times n^2$ and of strength $k < n$.*

Proof. Let $L(1,1)$ denote a $b \times n^2$ orthogonal array based on the symbols $1,2,3,\ldots,n$, where $b \geq 5$. Since any permutation applied to a row of an orthogonal array preserves orthogonality it may be assumed that each cell of column $(1,1)$ is occupied by 1. Let ρ denote a cycle on $\{1,2,3,\ldots,n\}$ and σ a cycle on $\{2,3,4,\ldots,n\}$. We can assume that σ is a permutation on $\{1,2,\ldots,n\}$ leaving 1 fixed. For each $i = 1,2,3,\ldots,k$, $L(1,i)$ is a $b \times n^2$ orthogonal array obtained from $L(1,1)$ by applying σ^{i-1} to row 3, row 4, and row 5, and leaving the other rows unchanged. Note that σ^0 is the identity permutation and thus $L(1,1)$ remains unchanged. For each $j = 1,2,3,\ldots,k$, the $b \times n^2$ orthogonal array $L(2,j)$ is obtained from $L(1,1)$ by applying ρ^j to row 3 of $L(1,1)$ and leaving the other rows unchanged.

Claim. The collection $L = \{L(i,j) \mid 1 \leq i \leq 2, 1 \leq j < k\}$ is a separating set of dimension $b \times n^2$ and strength k. Each $L(i,j)$ in L is obtained from $L(1,1)$ by applying permutations to the rows of $L(1,1)$ and thus inherits orthogonality from it.

(1) For every $i = 1,2,\ldots,k$, every cell of column $(1,1)$ of $L(1,i)$ is occupied by 1. This follows from the fact that σ leaves 1 fixed and therefore so does σ^{i-1}.

(2) For every $i, j \in \{1,2,\ldots,k\}$, $i \neq j$, $L(1,i)$ and $L(1,j)$ are disjoint on each column except column $(1,1)$. To see this, consider column $(\mu,\lambda) \neq (1,1)$ of $L(1,i)$ and $L(1,j)$, $i \neq j$. Let c_4 and c_5 denote the cells of column (μ,λ) on rows 4 and 5 respectively. Now if cell c_4 is occupied by the symbol x in both $L(1,i)$ and $(L(1,j)$, then $x = 1$. This follows from the fact that if $x \neq 1$, then $a\sigma^{i-1} = x = a\sigma^{j-1}$ where a is the entry in cell c_4 of $L(1,1)$. But σ is a cycle on $n-1$ symbols and $0 \leq i-1$, $j-1 \leq n-2$ so $i-1 = j-1$ and $i = j$, a contradiction. Similarly if cell c_5 is occupied by the same symbol in both $L(1,i)$ and $L(1,j)$, then that symbol must be 1. But 1 on the fourth and fifth rows of column (μ,λ) and column $(1,1)$ of $L(1,i)$ contradicts the fact that $L(1,i)$ is orthogonal. Thus at least one of cells c_4 and c_5 is occupied by distinct symbols in $L(1,i)$ and $L(1,j)$.

(3) For every $i, j \in \{1,2,\ldots,k\}$, $i \neq j$, $L(1,i)$ and $L(2,j)$ are disjoint. Consider column (μ,λ) of $L(1,i)$ and $L(2,j)$, $i \neq j$. Let c_3, c_4, and c_5 denote the cells of column (μ,λ) on rows 3, 4, and 5 respectively. Suppose $(\mu,\lambda) = (1,1)$. Since 1 is the

entry in cell c_3 of $L(1,i)$, $1\rho^i$ is the entry in cell c_3 of $L(2,j)$. Since ρ^i is a cycle on n symbols and $1 \leq j \leq n-1$, it follows that $1\rho^i \neq 1$ and thus $L(1,i)$ and $L(2,j)$ are disjoint on column (μ,λ). Now suppose $(\mu,\lambda) \neq (1,1)$. Since rows 4 and 5 of $L(2,j)$ are rows 4 and 5 of $L(1,1)$, at least one of cells c_4 and c_5 is occupied by distinct symbols in $L(1,i)$ and $L(2,j)$ by argument 2, so, $L(1,i)$ and $L(2,j)$ are disjoint on column (μ,λ). Thus $L(1,i)$ and $L(2,j)$ are disjoint.

4. Intersection preserving embedding of partial (n, b)-PBD's

Lemma 4.1. *If $b \geq 2$ and each of $(P,p_1),(P,p_2),\ldots,(P,p_k)$ is a partial $(|P|,b)$-PBD, then there is a collection $(V,t_1),(V,t_2),\ldots,(V,t_k)$ of $(|V|,b)$-PBD's such that for each i, $p_i \subseteq t_i$.*

Proof. There are numerous ways of doing this but the following is a generalization of the result of Rosa and Lindner in [5]. Let $X_1 = P, X_2,\ldots,X_k$ denote k mutually exclusive sets each containing $|P|$ elements. Let $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_k$ denote k bijective functions such that α_i maps P onto X_i. For each block a in p_i let $a\alpha_i = \{s\alpha_i \mid s$ is in $a\}$. Now it is easy to see that (Q,q) is a partial PBD all of whose blocks are of size b where $Q = X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_k$ and $q = \{a\alpha_i \mid a$ is in p_i and $1 \leq i \leq k\}$. Due to a remarkable theorem by Ganter [6], (Q,q) can be embedded in a PBD (V,t) whose blocks are all of size b. Since $|P| = |X_i|$, there is a bijective function β_i from $V\setminus X_i$ onto $V\setminus P$. Define γ_i by $s\gamma_i = s\alpha_i^{-1}$, if s is in X_i and $s\gamma_i = s\beta_i$, if s is in $V\setminus X_i$. Now γ_i is a permutation of the symbols of V that maps each block $a\alpha_i$ onto a. Thus (P,p_i) is embedded in (V,t_i) where $t_i = \{c\gamma_i \mid c$ is in $t\}$.

Lemma 4.2 (Chowla, Erdos, and Strauss [7]). *For each integer $b \geq 2$ there exists a constant $O(b)$ such that if $n > O(b)$, then there exists a collection of b mutually orthogonal latin squares of order n and thus a $(b+2)\times n^2$ orthogonal array and thus a $b \times n^2$ orthogonal array.*

Lemma 4.3 (Wilson [2]). *For each integer $b \geq 2$, there exists a constant $P(b)$ such that if $n > P(b)$ and $n-1 \equiv 0 \bmod (b-1)$ and $n(n-1) \equiv 0 \bmod (b(b-1))$, then there exists an (n,b)-PBD.*

Corollary 4.4. *If k and b are positive integers and $b \geq 5$, then there exists a constant $H(k,b)$ such that if $n > H(k,b)$, $n-1 \equiv 0 \bmod (b-1)$, and $n(n-1) \equiv 0 \bmod (b(b-1))$, then there exists a separating set of dimension $b \times n^2$ and strength k and a collection of k mutually disjoint (n,b)-PBD's $(Q,q_1),(Q,q_2),\ldots,(Q,q_k)$.*

By choosing $H(k,b)$ to be the maximum of $O(b)$, $P(b)$, and $C(k,b)$ this corollary is proved by Theorem 2.2, Theorem 3.5, Lemma 4.2 and Lemma 4.3.

Theorem 4.5. *If $(P, p_1), (P, p_2), \ldots, (P, p_k)$ are partial $(|P|, b)$-PBD's where $b \geq 3$, and n is an integer such that there exists a separating set L of dimension $b \times n^2$ and strength $k \geq 2$ collection of k mutually disjoint (n, b)-PBD's, then there exist k (q, b)-PBD's, $(X, x_1), (X, x_2), \ldots, (X, x_k)$ such that (P, p_i) is embedded in (X, x_i) and for $i \neq j$, $p_i \cap p_j = x_i \cap x_j$.*

Proof. Suppose the hypothesis is true and $(Q, q_1), (Q, q_2), \ldots, (Q, q_k)$ are k disjoint (n, b)-PBD's and that $P = \{1, 2, \ldots, p\}$. By Lemma 4.1, there exists a set $V = \{1, 2, \ldots, v\}$ and k (v, b)-PBD's $(V, t_1), (V, t_2), \ldots, (V, t_k)$ such that for each i, (P, p_i) is embedded in (V, t_i). Let $L = \{L(i, j) \mid 1 \leq i \leq 2 \text{ and } 1 \leq j \leq k\}$ as in the definition. Let us now form the generalized direct products $(Q \times V, T_1), (Q \times V, T_2), \ldots, (Q \times V, T_k)$, based on a construction in [5], as follows:

Type 1. $\{(a_1, x), (a_2, x), \ldots, (a_b, x)\}$ is in T_i for every $\{a_1, a_2, \ldots, a_b\}$ in q_i and x in V.

Type 2. $\{(a_1, x_1), (a_2, x_2), \ldots, (a_b, x_b)\}$ is in T_i provided that $t^* = \{x_1, x_2, \ldots, x_b\}$ is in t_i, $x_1 < x_2 < \cdots < x_b$, and $a_1, a_2, \ldots, a_b$ is column (a_1, a_2) of

$$\begin{cases} L(1, i), & \text{if } t^* \in p_i, \\ L(2, i), & \text{if } t^* \in t_i \setminus p_i. \end{cases}$$

Observation 4.6. $(Q \times V, T_i)$ is a (qv, b)-PBD.

Suppose $(a_1, x_1) \neq (a_2, x_2)$ are in $Q \times V$. If $x_1 = x_2$, then $\{(a_1, x_1), (a_2, x_2)\}$ is in exactly 1 Type-1 block and in no Type-2 block. If $x_1 \neq x_2$, then $\{(a_1, x_1), (a_2, x_2)\}$ is in no Type-1 block. There is a unique block $\{y_1, y_2, \ldots, y_b\} = t^*$ of t_i, $y_1 < y_2 < \cdots < y_b$, that contains $\{x_1, x_2\}$. Let $x_1 = y_\alpha$ and $x_2 = y_\beta$ where $\alpha \neq \beta$. Suppose t^* is in p_i. There exists a unique column (c_1, c_2) of $L(1, i)$ that contains a_1 on row α and a_2 on row β since $L(1, i)$ is an orthogonal array. Let $c_1, c_2, c_3, \ldots, c_b$ denote column (c_1, c_2) of $L(1, i)$. Thus $\{(c_1, y_1), (c_2, y_2), \ldots, (c_b, y_b)\}$ is in T_i and it contains (a_1, x_1) and (a_2, x_2). No other block of T_i based on $\{y_1, y_2, \ldots, y_b\}$ contains both (a_1, x_1) and (a_2, x_2) since this would imply the existence of a column different from column (c_1, c_2) in $L(1, i)$ having a_1 on row α and a_2 on row β. Since every other Type-2 block is based on a block of t_i that is different from t^* and t^* is the only one that contains both x_1 and x_2, the only block of T_i containing $\{(a_1, x_1), (a_2, x_2)\}$ is $\{(c_1, y_1), (c_2, y_2), \ldots, (c_b, y_b)\}$. The argument for t^* not in p_i is the same as the above argument except substitute $L(2, i)$ for $L(1, i)$.

Observation 4.7. If $\{x_1, x_2, \ldots, x_b\}$ is in p_i, then $\{(1, x_1), (1, x_2), \ldots, (1, x_b)\}$ is in T_i. This follows from the fact that column $(1, 1)$ of $L(1, i)$ is all 1's.

Observation 4.8. For every $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, let

$$S(i, j) = \{\{(1, a_1), (1, a_2), \ldots, (1, a_b)\} \mid \{a_1, a_2, \ldots, a_b\} \in P_i \cap P_j\}.$$

It follows from Observation 4.7 that $S(i, j) \subseteq T_i \cap T_j$.

Observation 4.9. $T_i \cap T_j \subseteq S(i,j)$ and thus from Observation 4.8, $T_i \cap T_j = S(i,j)$.

Suppose $r = \{(a_1, x_1), (a_2, x_2), \ldots, (a_b, x_b)\} \in T_i \cap T_j$. Since (Q, q_i) and (Q, q_j) are disjoint, T_i and T_j have no Type-1 block in common, thus r must be of Type-2 in both T_i and T_j. By construction $\{x_1, x_2, \ldots, x_b\} = t^* \in t_i$ and $t^* \in t_j$. Without loss of generality we may assume that $x_1 < x_2 < \cdots < x_b$. Now $a_1, a_2, \ldots, a_b$ is column (a_1, a_2) of exactly one of $L(1,i)$ and $L(2,i)$; additionally $a_1, a_2, \ldots, a_b$ is column (a_1, a_2) of exactly one of $L(1,j)$ and $L(2,j)$. Since $L(1,i)$ and $L(2,j)$ are disjoint, $L(2,i)$ and $L(2,j)$ are disjoint, and $L(2,i)$ and $L(1,j)$ are disjoint, $a_1, a_2, \ldots, a_b$ must be column (a_1, a_2) of $L(1,i)$ and $L(1,j)$. Since $L(1,i)$ and $L(1,j)$ are disjoint on each column except column $(1,1)$, column (a_1, a_2) is column $(1,1)$, $a_1, a_2, \ldots, a_b = 1, 1, \ldots, 1$, and $r \in S(i,j)$. Therefore $T_i \cap T_j \subseteq S(i,j)$ and $T_i \cap T_j = S(i,j)$.

Now rename the symbols of $Q \times V$ in such a way that $(1, x)$ is named x for each x in P and the other symbols of $Q \times V$ become $p+1, p+2, \ldots, nv$. Let $X = \{1, 2, \ldots, nv\}$ and let x_i denote the collection of blocks derived from T_i by renaming the symbols in each block. Note that $S(i,j)$ becomes $p_i \cap p_j$ and that $T_i \cap T_j$ becomes $x_i \cap x_j$. Thus (P, p_i) is embedded in (X, x_i) and $i \neq j$ then $p_i \cap p_j = x_i \cap x_j$. This completes the proof of Theorem 4.5.

Theorem 4.10. *Suppose k and b are positive integers, $k \geq 2$ and $b \geq 5$. If $(P, p_1), (P, p_2), \ldots, (P, p_k)$ is a collection of partial $(|P|, b)$-PBD's, there exists k (n, b)-PBD's $(X, x_1), (X, x_2), \ldots, (X, x_k)$ such that (P, p_i) is embedded in (X, x_i) and for $i \neq j$, $p_i \cap p_j = x_i \cap x_j$.*

This theorem is an obvious result of Corollary 4.4 and Theorem 4.5.

5. Problem

In attacking the problem for $b = 4$, one needs k mutually disjunct $(n, 4)$-PBD's and a separating set of dimension $4 \times n^2$ and strength k for some positive integer n. The construction for a separating set in this paper obviously requires that $b \geq 5$, although using a generalization of Lindner and Rosa's technique in [5], one can construct a separating set of dimension $4 \times n^2$ and strength k for an infinite class of positive integers n for which an $(n, 4)$-PBD exists, although by no means all such n. So the real problem for $b = 4$ is generating large classes of mutually disjoint $(n, 4)$-PBD's. Naturally if Lemma 2.1 were true for $b = 4$ and all k then the embedding discussed in this paper could be done for $b = 4$. But Lemma 2.1 is false if $b = 4$ and $k \geq 6$, although Lemma 2.1 is true for $k = 1, 2, \ldots, 5$ (choose $C(k, 4) = 23$). Thus Theorem 2.2 is true for $b = 4$ and $1 \leq k \leq 5$ but not necessarily for $k \geq 6$. I do believe that if $k \geq 6$ there is some constant $C(k, 4)$ for which Theorem 2.2 is true. If such is the case Theorem 4.10 can be extended to $b = 4$.

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